APPLICATION OF METHOD OF LOCAL POTENTIAL TO THE BÉNARD PROBLEM IN A UNIFORM MAGNETIC FIELD

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The method of local potential is used to solve the Bénard problem in a uniform magnetic field in the case when the two boundary surfaces are free. The results obtained are in agreement with those known from the literature.

The thermodynamic theory of stability, possessing a considerable degree of generality (which enables different systems to be described from unified positions), distinguishes two approaches: the first connected with Lyapunov's second method [1], when the characteristic function is taken to be the deviation of the entropy from its value in the stationary state $(\delta^2 S)_0$ [2, 3]; the second, the method of local potential [3-5], connected with the method of characteristic indices. The latter method is used by us to investigate the Bénard problem in a uniform magnetic field.

We consider a thin layer of incompressible, conducting, electrically neutral fluid of thickness d heated from below and situated in gravitational and uniform magnetic fields. We assume that the fluid is nonpolarizable and that the displacement current can be neglected. Introducing the dimensionless quantities

$$x_i = \frac{\bar{x}_i}{d};\quad V_i = V_i\frac{d}{\nu};\quad t = \bar{t}\frac{\nu}{d^2};\quad T = \frac{\bar{T}}{\Delta T};\quad H_i = \frac{\bar{H}_i}{H^+}, \tag{1}$$

we obtain using Maxwell's equations the following expressions for the balance of mass, momentum, magnetic field intensity, and heat transport in the Boussinesq approximation:

$$\frac{\partial V_i}{\partial x_i} = 0,$$

$$P_1\frac{\partial V_i}{\partial t} + P_1 V_j\frac{\partial V_i}{\partial x_j} - \frac{P_1}{P_2}QH_j\frac{\partial H_i}{\partial x_j} = \frac{-\partial}{\partial x_i}\left(P + \frac{P_1}{P_2}Q\frac{H^2}{2}\right) + P_1\nabla^2 V_i + [1 - \alpha\Delta T(T - T^0)]F_i\frac{d^3}{\varkappa\nu}, \tag{2}$$

$$P_1\frac{\partial H_i}{\partial t} = P_1 H_j\frac{\partial V_i}{\partial x_j} + \frac{P_1}{P_2}\nabla^2 H_i,$$

$$P_1\frac{\partial T}{\partial t} = \nabla^2 T - P_1 V_j\frac{\partial T}{\partial x_j},$$

where

$$P_1 = \frac{\nu}{\varkappa};\quad P_2 = \frac{\nu}{\eta} = 4\pi\mu\sigma\nu;\quad Q = \frac{\mu H^2 d^2}{4\pi\rho\nu\eta};\quad H^+ = H_z.$$

We investigate the stability of the steady state of a fixed layer of fluid situated between two surfaces at constant temperatures T^0 and T_1 in a uniform magnetic field directed along the z axis. The steady state is described by the equations

$$V_i^{st} = 0;\quad H_i^{st} = \lambda_i;\quad T^{st} = T^0 - Z;$$
$$\frac{\rho^{st}}{\rho^0} = [1 - \alpha\Delta T(T^{st} - T^0)], \tag{3}$$

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where T^0 and ρ^0 denote the temperature and density on the lower boundary, and $\vec{\lambda} = (0, 0, 1)$. Setting

$$V_i = U_i, \quad H_i = H_i^{st} + h_i, \quad T = T^{st} + \theta, \\ \rho = \rho^{st} + \delta\rho, \quad p = p^{st} + \delta p, \tag{4}$$

we obtain the system of perturbed equations

$$\frac{\partial U_i}{\partial x_i} = 0,$$
$$P_1 \frac{\partial U_i}{\partial t} = -\frac{\partial}{\partial x_i}\left(\delta p + \frac{P_1}{P_2} Q\lambda_i h_i\right) + R\lambda_i\theta + P_1\nabla^2 U_i + \frac{P_1}{P_2} Q\lambda_j \frac{\partial h_i}{\partial x_j},$$
$$P_1 \frac{\partial \theta}{\partial t} - P_1 U_i \lambda_i = \nabla^2\theta, \tag{5}$$
$$P_1 \frac{\partial h_i}{\partial t} = P_1\lambda_j \frac{\partial U_i}{\partial x_j} + \frac{P_1}{P_2}\nabla^2 h_i,$$

where

$$R = \frac{d^3 \Delta T \alpha g}{\varkappa \nu}.$$

Applying the curl operator twice to the second equation and multiplying by λ_i, we arrive at

$$P_1 \frac{\partial}{\partial t}\nabla^2 U_z = R\left(\frac{\partial^2\theta}{\partial x^2} + \frac{\partial^2\theta}{\partial y^2}\right) + P_1\nabla^4 U_z + \frac{P_1}{P_2} Q \frac{\partial}{\partial z}\nabla^2 h_z,$$
$$P_1 \frac{\partial \theta}{\partial t} - P_1 U_z = \nabla^2\theta, \tag{6}$$
$$P_1 \frac{\partial h_z}{\partial t} = P_1 \frac{\partial U_z}{\partial z} + \frac{P_1}{P_2}\nabla^2 h_z.$$

Introducing into (6) the perturbations

$$U_z = W(z)\cos k_x x \cos k_y y \exp\beta t,$$
$$h_z = h(z)\cos k_x x \cos k_y y \exp\beta t,$$
$$\theta = \theta(z)\cos k_x x \cos k_y y \exp\beta t$$

gives the equations

$$P_1 (D^2 - k^2)(D^2 - k^2 - \beta) W + \frac{P_1}{P_2} Q D (D^2 - k^2) h = k^2 R\theta, \\ (D^2 - k^2 - P_1\beta)\theta = -P_1 W, \quad (D^2 - k^2 - P_2\beta) h = -P_2 DW \tag{7}$$

where

$$D^n = \frac{d^n}{dz^n}, \quad k^2 = k_x^2 + k_y^2,$$

whose solutions must satisfy the boundary conditions

$$W = \theta = 0 \text{ for } z = 0, \ z = 1$$

and

$$DW = 0 \text{ on a rigid boundary,}$$
$$D^2 W = 0 \text{ on a free boundary,}$$

and h is continuous with the external field for a nonconducting boundary and equals zero on a perfectly conducting surface.

The stationary state will be unstable if $\mathrm{Re}\beta > 0$, and stable if $\mathrm{Re}\beta < 0$. The case $\mathrm{Re}\beta = 0$ determines a critical state separating regions of stability and instability. If $\mathrm{Im}\beta = 0$ (principle of change of stability satisfied), the critical state is stationary; otherwise, it is oscillating (superstability).

If the principle of change of stability holds, we obtain, setting $\beta = 0$ in (7),

$$P_1 (D^2 - k^2)^2 W + \frac{P_1}{P_2} QD (D^2 - k^2) h = k^2 R\theta,$$
$$(D^2 - k^2)\theta = -P_1 W, \tag{8}$$
$$(D^2 - k^2) h = -P_2 DW.$$

The structure of the equations of system (8) is such that the number of unknown functions can be reduced by eliminating some of them (for example, h or θ, h). Thus, by inserting the latter equation of system (8) into the first or by using the operator $(D^2 - k^2)$, allowing for the second and third equations, we obtain

$$P_1 [(D^2 - k^2)^2 - QD^2] W = k^2 R\theta, \ (D^2 - k^2)\theta = -P_1 W \tag{9}$$

and

$$(D^2 - k^2) [(D^2 - k^2)^2 - QD^2] W = -Rk^2 W \tag{10}$$

with corresponding boundary conditions.

In many cases it is very difficult and even impossible to find the exact eigenfunctions and frequencies, and for this reason it is advantageous to employ the variational technique of local potential [3, 5]. In order to obtain the local potential we multiply the second, third, and fourth equations in (2) by $-\delta U_i$, $-\delta\theta$, $-\delta h_i$, add, and integrate by parts terms containing δU_i and $\delta\theta$ remembering that

$$\frac{\partial A_i}{\partial t}\delta A_i = \frac{\partial}{\partial t}\frac{(\delta A_i)^2}{2} - \frac{\partial}{\partial t} A_i^0 \delta A_i$$

(A_i is a scalar, or the component of a vector or tensor) and that δU_i and $\delta\theta$ equal zero on the boundary. We then find that

$$\Phi = \int dv \left\{ -\left(\delta p^0 + \frac{P_1}{P_2} Q\lambda_i h_i\right)\frac{\partial U_i}{\partial x_i} - R\theta^0 \lambda_i U_i + \right.$$
$$+ \frac{P_1}{2}\left(\frac{\partial U_i}{\partial x_j}\right)^2 + \frac{P_1}{P_2} Q\lambda_i h_i^0 \frac{\partial U_i}{\partial x_j} + \frac{1}{2}\left(\frac{\partial\theta}{\partial x_i}\right)^2 -$$
$$\left. - \frac{P_1}{P_2}(\nabla^2 h_i^0) h_i - P_1 \lambda_j U_i^0 \frac{\partial h_i}{\partial x_j} + P_1\left[\frac{\partial U_i^0}{\partial t} U_i + \frac{\partial\theta^0}{\partial t}\theta + \frac{\partial h_i^0}{\partial t} h_i\right]\right\}. \tag{11}$$

The potential Φ depends on two types of variable: fluctuating and nonvarying. The latter we indicate by the superscript 0.

It is evident that $\delta\Phi = 0$ for additional conditions leads to the system of equations (2) and $\Delta\Phi > 0$. Accordingly [3], Φ is a suitable potential of our problem.

Choosing

$$U_z = W(z)\cos k_x x \cos k_y y \exp \beta t;$$
$$h_z = h(z)\cos k_x x \cos k_y y \exp \beta t;$$
$$U_x = -\frac{k_x}{k^2}(DW)\cos k_y y \sin k_x x \exp \beta t;$$
$$h_x = -\frac{k_x}{k^2}(Dh)\sin k_x x \cos k_y y \exp \beta t; \tag{12}$$
$$U_y = -\frac{k_y}{k^2}(DW)\cos k_x x \sin k_y y \exp \beta t;$$
$$h_y = -\frac{k_y}{k^2}(Dh)\cos k_x x \sin k_y y \exp \beta t;$$
$$\theta = \theta(z)\cos k_x x \cos k_y y \exp \beta t$$

(similar expressions are chosen for the nonvarying quantities) and substituting into (11), we obtain the potentials

$$\Phi = \int dz \left\{ -R\theta^0 W + \frac{P_1}{2}\left[k^2 W^2 + 2(DW)^2 + \frac{1}{k^2}(D^2 W)^2\right] + \right.$$

$$+\frac{P_1}{P_2} Q\left[h^0-\frac{1}{k^2}(D^2 h_0)\right](DW)+\frac{1}{2}[(D\theta)^2+k^2\theta^2]-$$

$$-\frac{P_1}{P_2}\left[h(D^2-k^2)h^0+\frac{1}{k^2}(Dh)(D^2-k^2)(Dh^0)\right]-P_1W^0\theta+$$

$$+P_1\left[(D^2-k^2)W^0(Dh)+\frac{1}{k^2}(DW^0)(D^2h)\right]+$$

$$+P_1\beta^0\left[WW^0+\theta\theta^0+hh^0+\frac{1}{k^2}(DW)(DW^0)+\frac{1}{k^2}(Dh^0)(Dh)\right]\Bigg\} \tag{13}$$

and

$$\Phi^1=\int dz\Bigg\{-R\theta^0W+\frac{P_1}{2}\left[k^2W^2+2(DW)^2+\frac{1}{k^2}(D^2W)^2\right]+$$

$$+\frac{P_1}{P_2}Q\left[h^0-\frac{1}{k^2}(D^2h^0)\right](DW)+\frac{1}{2}[(D\theta)^2+k^2\theta^2]-$$

$$-P_1W^0\theta-\frac{P_1}{P_2}\left[h(D^2-k^2)h^0+\frac{1}{k^2}(Dh)(D^2-k^2)(Dh^0)\right]+P_1\left[W^0(Dh)+\frac{1}{k^2}(DW^0)(D^2h)\right]\Bigg\}, \tag{14}$$

by the Euler–Lagrange equations, given by the system of equations (7), (8) for the additional conditions $A_i = A_i^0$. In the same manner as above it is possible to obtain potentials that depend on a smaller number of unknown functions. Thus, for example, on making the substitution

$$A_i^0 \to (D^2-k^2-P_2\beta)A_i^0,$$
$$A_iA_i \to A_i(D^2-k^2-P_2\beta)A_i$$

in terms containing W and taking account of the third equation of (7), or making the substitution

$$A_i^0 \to (D^2-k^2-P_2\beta)(D^2-k^2-P_1\beta)A_i^0,$$
$$A_iA_i \to A_i(D^2-k^2-P_2\beta)(D^2-k^2-P_1\beta)A_i$$

and taking account of the second and third equations of (7), it is possible to convert (13) to

$$\Phi_1=\int dz\Bigg\{-R(D^2-k^2-P_2\beta)\theta^0W+\frac{P_1}{2}\Big[k^2W(D^2-k^2-P_2\beta)W+$$

$$+2(DW)(D^2-k^2-P_2\beta)(DW)+\frac{1}{k^2}(D^2W)(D^2-k^2-P_2\beta)(D^2W)\Big]-$$

$$-P_1Q\left[(DW^0)-\frac{1}{k^2}(D^3W^0)\right](DW)+\frac{1}{2}[(D\theta)^2+k^2\theta^2]-$$

$$-P_1\beta^0\left[W(D^2-k^2-P_2\beta)W^0+\frac{1}{k^2}(DW)(D^2-k^2-P_2\beta)(DW^0)+\theta\theta^0\right]\Bigg\} \tag{15}$$

and

$$\Phi_2=\int dz\Bigg\{-RW(D^2-k^2-P_2\beta)P_1W^0+\frac{P_1}{2}\Big[k^2W(D^2-k^2-P_2\beta)\times$$

$$\times(D^2-k^2-P_1\beta)W+\frac{1}{k^2}(D^2W)(D^2-k^2-P_2\beta)(D^2-k^2-P_1\beta)(D^2W)+$$

$$+2(DW)(D^2-k^2-P_2\beta)(D^2-k^2-P_1\beta)(DW)\Big]-P_1Q(DW)\times$$

$$\times\left[(D^2-k^2-P_1\beta)(DW^0)-\frac{1}{k^2}(D^2-k^2-P_1\beta)(D^3W^0)\right]+P_1\beta^0\Big[W(D^2-$$

$$-k^2-P_2\beta)(D^2-k^2-P_1\beta)W^0+\frac{1}{k^2}(DW)(D^2-k^2-P_2\beta)(D^2-k^2-P_1\beta)(DW^0)\Big]\Bigg\}, \tag{16}$$

which depend on two and one unknown functions, respectively. Trial functions are chosen depending on the type of boundary surfaces.

Let us consider the case when both boundary surfaces are free.

a) Appearance of Instability as Stationary Convection. Substitution of the third equation of (8) into (14) leads to the expression

$$\Phi_1^1 = \int dz \left\{ -R\theta^0 W + \frac{P_1}{2}\left[k^2W^2 + 2(DW)^2 + \frac{1}{k^2}(D^2W)^2\right] + \right.$$
$$\left. + \frac{P_1}{k^2} Q (DW)(DW^0) + \frac{1}{2}[(D\theta)^2 + k^2\theta^2] - P_1 W^0\theta \right\}, \tag{17}$$

which is the potential of problem (9).

We prescribe the unknown functions in the form

$$W = A\sin\pi z;\ \theta = B\sin\pi z;\ W^0 = A^0\sin\pi z;\ \theta^0 = B^0\sin\pi z, \tag{18}$$

where the constants A_i, B_i are determined from the condition

$$\begin{aligned} &\left(\frac{\partial\Phi}{\partial A_i}\right)_{B_i^0,\,A_i^0,\,B_i} = 0, && A_i^0 = A_i, \\ & && B_i^0 = B_i, \\ &\left(\frac{\partial\Phi}{\partial B_i}\right)_{A_i,\,B_i^0,\,A_i^0} = 0, && \beta^0 = \beta. \end{aligned} \tag{19}$$

In this manner we obtain

$$\begin{aligned} &\frac{P_1}{k^2}[k^4 + 2\pi^2k^2 + \pi^4 + Q\pi^2]A - RB = 0, \\ &-P_1A + (\pi^2 + k^2)B = 0, \end{aligned} \tag{20}$$

which gives

$$R = \frac{(\pi^2 + k^2)}{k^2}[(\pi^2 + k^2)^2 + \pi^2 Q]. \tag{21}$$

In a similar manner potential (14) and trial functions

$$W = A\sin\pi z;\quad \theta = B\sin\pi z;\quad h = C\cos\pi z \tag{22}$$

with conditions (19) give

$$\begin{aligned} &\frac{P_1}{k^2}(\pi^2 + k^2)A - RB + \frac{P_1}{P_2}\frac{Q}{k^2}\pi(\pi^2 + k^2)C = 0, \\ &-P_1A + (\pi^2 + k^2)B = 0,\quad -P_2\pi A + (\pi^2 + k^2)C = 0, \end{aligned} \tag{23}$$

which reduces to (21).

b) Appearance of Instability as Oscillating Convection. Utilizing (13) and perturbations (22), we arrive at

$$\begin{aligned} &\frac{P_1}{k^2}(\pi^2 + k^2 + \beta)(\pi^2 + k^2)A - RB + \frac{P_1}{P_2}Q\frac{\pi}{k^2}(\pi^2 + k^2)C = 0, \\ &-P_1A + (\pi^2 + k^2 + P_1\beta)B = 0, \\ &-P_2\pi A + (k^2 + \pi^2 + P_2\beta)C = 0, \end{aligned} \tag{24}$$

from which we find

$$(k^2 + \pi^2 + P_1\beta)(\pi^2 + k^2)[(k^2 + \pi^2 + P_2\beta)(k^2 + \pi^2 + \beta) + \pi^2 Q] = Rk^2(\pi^2 + k^2 + P_2\beta). \tag{25}$$

In a similar manner potential (15) and perturbations (18) or (16) and $W = A\sin \Pi z$ again give (25).

It follows from (25) [6] that for oscillating solutions to exist the condition $\varkappa > \eta$ must be satisfied. Then, for given P_1 and P_2, superstability is possible only if Q is greater than a certain Q^1.

The results obtained by the thermodynamic method using various local potentials thus coincide among themselves and with the exact solution, which is connected with the fortunate choice of the trial functions.

The use of potentials with a smaller number of unknown functions is to be preferred since this reduces the degree of arbitrariness, which increases as the number of functions being varied increases, and enables a better choice to be made as the amount of information on the conditions imposed on them accordingly also increases. Furthermore, the possibility of increasing the number of variable parameters arises, which makes the trial functions more flexible without noticeably increasing the volume of computations.

The method continues to work quite effectively in those cases when the number of unknown functions cannot be reduced, although perhaps slightly less accurately.

NOTATION

V_i, H_i, components of velocity of center of mass and magnetic field intensity; ρ, density; p, pressure; μ, magnetic susceptibility; T, temperature; $\varkappa$, thermal diffusivity; σ, electrical conductivity; α, coefficient of thermal expansion; $\vec{\lambda}$, vector with components (0, 0, 1); ∇, Hamiltonian operator; ΔT, temperature drop between boundaries.

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EFFECTS OF THERMOELASTIC STRESS DURING HONING ON COMPONENT DIMENSIONS

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Equations are given for the temperature distribution and thermoelastic deformations arising in the workpiece and tool during honing, as well as a method for determining the corrections for the temperature errors in adjusting automatic-control devices.

The uses of honing are extending continually; when the process is introduced, it is usually necessary to provide automatic monitoring of workpiece dimensions.

In turn, the setting-up procedure for the monitoring requires information about the errors arising from thermal expansion of the workpiece and the tool, since the latter can be very important, especially for thin-walled components.

The thermal deformation can be determined from the temperature distribution produced in the workpiece 1 and honing rod 2 (Fig. 1), with subsequent calculation of the deformation on the basis of the theory of elasticity.

The hone rotates at a relatively high speed, and the area of contact with the workpiece is fairly large, so one can assume that the temperature over the entire inner surface of the component is the same to a first approximation and equal to the contact temperature θ_c. We may also suppose that the outer surface A of the workpiece 1 (Fig. 1) has boundary conditions of the third kind, since this surface is usually cooled by a continuous flow of liquid.

The temperature of the coolant (kerosene) and the initial temperature of the workpiece may be taken as nominally zero.

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